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Some new regularity criteria for the 3D MHD equations

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ABSTRACT

In this paper we consider the regularity criteria for solutions to the 3D incompressible magnetohydrodynamics (MHD) equations. The first one involves $\nabla_h u = (\partial_1 u, \partial_2 u)$ and $\partial_3 b$. Then, the criterion concerning $\nabla_h u$ and $\nabla_h b = (\partial_1 b, \partial_2 b)$ is presented. Finally, we show that the weak solution (u, b) is actually regular, provided that u_3, $\partial_3 u$, b and $\partial_3 b$ are in suitable Serrin-type integrability classes, respectively.

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1. Introduction

We consider the following 3D MHD equations in this paper:

$$\begin{cases} \partial_t u + u \cdot \nabla u = \Delta u - \nabla p - \dfrac{1}{2}\nabla |b|^2 + b \cdot \nabla b + f, \\ \partial_t b + u \cdot \nabla b = \Delta b + b \cdot \nabla u + g, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ u(x,0) = u_0(x), \qquad b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where: $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field; $b \in \mathbb{R}^3$ is the magnetic field; $p(x,t)$ is a scalar pressure; f represents volume force applied to the fluid; g is usually zero when Maxwell's displacement currents are ignored; $u_0(x)$ with $\operatorname{div} u_0 = 0$ is the initial velocity field. In this paper, we assume that $f = g = 0$, just for simplicity.

The study of the incompressible MHD equations (1.1) in 3D space has a long history. In the pioneering work [1], Semange and Teman proved the local well-posedness of weak solutions for any given initial datum $u_0, b_0 \in H^s(\mathbb{R}^3), s \geq 3$. But whether this unique local solution can exist globally is an outstanding challenging problem. Fundamental Serrin-type regularity criteria only in terms of the velocity were given in [2,3] independently. Recently, some improvement and extension was achieved on the basis of these two basic papers. Parts of them are listed here: Chen et al. [4] proved regularity by adding a condition on $\Delta_j(\nabla \times u)$. He and Wang [5] extended it to the weak L^p-space $L^{p,\infty}$. Zhou and Gala [6] proved regularity for u and ∇u in the multiplier spaces. Chen et al. [7] showed the regularity in terms of the direction of the velocity. Wu [8] considered the velocity field being in the homogeneous Besov space. Regularity was obtained by imposing a condition on the pressure in [9] and regularity criteria in terms of the vorticity field and its direction were established in [10,11] (see also [2]) respectively. Recently, logarithmically improved regularity criteria for the MHD equations were established in [12].

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Recently, some regularity criteria in terms of partial velocity components, the derivative of the velocity field and the pressure were established in [13–16]. But the spaces used are not scaling invariant (in other words, not of Serrin's type).

The purpose of this paper is to establish some new regularity criteria of weak solutions in scaling invariant spaces. Before going to the main theorems, we introduce some notation. For convenience, we will use the symbol ∇_h for (∂_1, ∂_2). We also write $L^{p,q}$ for $L^p(0, T; L^q(\mathbb{R}^3))$.

The first theorem reads:

Theorem 1.1. *Assume that the initial velocity and magnetic fields* $(u_0, b_0) \in H^s(\mathbb{R}^3)$ *with* $s \geq 3$, $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Additionally, assume that*

$$\begin{aligned} &\nabla_h u \in L^{\alpha_1,\gamma_1}, \quad \text{with } \frac{2}{\alpha_1} + \frac{3}{\gamma_1} \leq 2, \frac{3}{2} < \gamma_1 \leq \infty, \\ &\partial_3 b \in L^{\alpha_2,\gamma_2}, \quad \text{with } \frac{2}{\alpha_2} + \frac{3}{\gamma_2} \leq 2, \frac{3}{2} < \gamma_2 \leq \infty, \end{aligned} \tag{1.2}$$

or both $\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}}$ *and* $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}$ *are sufficiently small on* $[0, T]$*; then the corresponding solution remains smooth on* $[0, T]$.

A natural question is whether we can replace $\partial_3 b$ by $\partial_1 b$ or $\partial_2 b$ in Theorem 1.1. Unfortunately, this is impossible. However, the following result holds where $\partial_3 b$ in Theorem 1.1 is replaced by $\nabla_h b$.

Theorem 1.2. *Suppose that the initial velocity and magnetic fields* $(u_0, b_0) \in H^s(\mathbb{R}^3)$ *with* $s \geq 3$, $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Additionally, assume that*

$$\begin{aligned} &\nabla_h u \in L^{\alpha_1,\gamma_1}, \quad \text{with } \frac{2}{\alpha_1} + \frac{3}{\gamma_1} \leq 2, \frac{3}{2} < \gamma_1 \leq \infty, \\ &\nabla_h b \in L^{\alpha_2,\gamma_2}, \quad \text{with } \frac{2}{\alpha_2} + \frac{3}{\gamma_2} \leq 2, \frac{3}{2} < \gamma_2 \leq \infty, \end{aligned}$$

or both $\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}}$ *and* $\|\nabla_h b\|_{L^{\infty,\frac{3}{2}}}$ *are sufficiently small on* $[0, T]$*; then the corresponding solution* $(u(x, t), b(x, t))$ *remains smooth on* $[0, T]$.

The third main result is as follows.

Theorem 1.3. *Assume that the initial velocity and magnetic fields* $(u_0, b_0) \in H^s(\mathbb{R}^3)$ *with* $s \geq 3$, $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. *Additionally, assume that*

$$\begin{aligned} &u_3 \in L^{\alpha_1,\gamma_1}, \quad \text{with } \frac{2}{\alpha_1} + \frac{3}{\gamma_1} \leq 1, 3 < \gamma_1 \leq \infty, \\ &\partial_3 u \in L^{\alpha_2,\gamma_2}, \quad \text{with } \frac{2}{\alpha_2} + \frac{3}{\gamma_2} \leq 2, \frac{3}{2} < \gamma_2 \leq \infty, \\ &b \in L^{\alpha_3,\gamma_3}, \quad \text{with } \frac{2}{\alpha_3} + \frac{3}{\gamma_3} \leq 1, 3 < \gamma_3 \leq \infty, \\ &\partial_3 b \in L^{\alpha_4,\gamma_4}, \quad \text{with } \frac{2}{\alpha_4} + \frac{3}{\gamma_4} \leq 2, \frac{3}{2} < \gamma_4 \leq \infty, \end{aligned}$$

or $\|u_3\|_{L^{\infty,3}}$, $\|\partial_3 u\|_{L^{\infty,\frac{3}{2}}}$, $\|b\|_{L^{\infty,3}}$ *and* $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}$ *are sufficiently small on* $[0, T]$*; then the corresponding solution* $(u(x, t), b(x, t))$ *remains smooth on* $[0, T]$.

The rest of this paper is organized as follows. In Section 2, we will show Theorem 1.1. Then the proof of Theorem 1.2 will be presented in Section 3. In the last section, we will prove Theorem 1.3. For simplicity, we drop $\mathbb{R}^3$ in our notation for function spaces if there is no ambiguity. Additionally, $\|\cdot\|_s$ $(0 < s \leq \infty)$ denotes the Sobolev norm of $L^s(\mathbb{R}^3)$ in this paper.

2. Proof for Theorem 1.1

To prove Theorem 1.1, it is sufficient to show

$$(u, b) \in L^\infty\left(0, T; H^1\right) \cap L^2\left(0, T; H^2\right), \tag{2.1}$$

if (1.2) holds.

Proof. Multiplying the first equation in (1.1) by Δu, after integration by parts and taking the divergence free property into account, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|\Delta u\|_2^2 = -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j dx - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_k\cdot\partial_i\partial_k u_j\cdot\partial_i b_j dx. \tag{2.2}$$

Similarly, multiplying the second equation of (1.1) by Δb, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla b\|_2^2 + \|\Delta b\|_2^2 = -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} b_k\cdot\partial_k\partial_i u_j\cdot\partial_i b_j dx. \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) + \left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right)\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k u_j\cdot\partial_i u_j dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j dx\\
&\quad - \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_k\cdot\partial_k b_j\cdot\partial_i b_j dx + \sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k u_j\cdot\partial_i b_j dx\\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{2.4}$$

First, considering the first term I_1 in (2.4), we have

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^3} u\cdot\nabla u\cdot\Delta u dx = \sum_{i=1}^{2}\int_{\mathbb{R}^3} u_i\cdot\partial_i u\cdot\Delta u dx + \int_{\mathbb{R}^3} u_3\cdot\partial_3 u\cdot\Delta u dx\\
&\equiv I_{11} + I_{12}.
\end{aligned} \tag{2.5}$$

Now we estimate the above terms one by one. First, we divide I_{11} into three parts, obtaining

$$\begin{aligned}
I_{11} &= -\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_j u_i\cdot\partial_i u\cdot\partial_j u dx - \sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3} u_i\cdot\partial_j\partial_i u\cdot\partial_j u dx\\
&= -\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_j u_i\cdot\partial_i u\cdot\partial_j u dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u_i\left(\partial_j u\right)^2 dx\\
&= -\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_i\cdot\partial_i u\cdot\partial_j u dx - \sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_3 u_i\cdot\partial_i u\cdot\partial_3 u dx + \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u_i\left(\partial_j u\right)^2 dx\\
&\equiv I_{111} + I_{112} + I_{113}.
\end{aligned} \tag{2.6}$$

For the first term, we know that

$$\begin{aligned}
|I_{111}| &\le \|\nabla_h u\|_{\gamma_1}\|\nabla_h u\|_{\frac{2\gamma_1}{\gamma_1-1}}^2 \le C\|\nabla_h u\|_{\gamma_1}\|\nabla u\|_2^{\frac{2\gamma_1-3}{\gamma_1}}\|\Delta u\|_2^{\frac{3}{\gamma_1}}\\
&\le \frac{1}{32}\|\Delta u\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla u\|_2^2.
\end{aligned}$$

Using the same method, we obtain that

$$|I_{112} + I_{113}| \le \|\nabla_h u\|_{\gamma_1}\|\nabla u\|_{\frac{2\gamma_1}{\gamma_1-1}}^2 \le \frac{1}{32}\|\Delta u\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla u\|_2^2.$$

Combining the two inequalities with (2.6) yields

$$|I_{11}| \leq \frac{1}{16}\|\Delta u\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla u\|_2^2. \tag{2.7}$$

Then, we consider the second term; it follows that

$$\begin{aligned} I_{12} &= -\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_j u_3\cdot\partial_3 u\cdot\partial_j u dx - \sum_{j=1}^{3}\int_{\mathbb{R}^3} u_3\cdot\partial_j\partial_3 u\cdot\partial_j u dx \\ &= -\sum_{j=1}^{3}\partial_j u_3\cdot\partial_3 u\cdot\partial_j u dx + \frac{1}{2}\sum_{j=1}^{3}\partial_3 u_3\cdot\left(\partial_j u\right)^2 dx \\ &= -\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_3\cdot\partial_3 u\cdot\partial_j u dx - \int_{\mathbb{R}^3}\partial_3 u_3\cdot(\partial_3 u)^2 dx + \frac{1}{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3}\partial_3 u_3\cdot\left(\partial_j u\right)^2 dx \\ &= -\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_3\cdot\partial_3 u\cdot\partial_j u dx + \sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u_i\cdot(\partial_3 u)^2 dx - \frac{1}{2}\sum_{j=1}^{3}\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u_i\cdot\left(\partial_j u\right)^2 dx \\ &\equiv I_{121} + I_{122} + I_{123}, \end{aligned}$$

which implies that

$$|I_{12}| \leq |I_{121}| + |I_{122}| + |I_{123}| \leq \frac{1}{16}\|\Delta u\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla u\|_2^2, \tag{2.8}$$

where we use estimates similar to those used before. Putting (2.7) and (2.8) into (2.5) we obtain

$$|I_1| \leq \frac{1}{8}\|\Delta u\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla u\|_2^2. \tag{2.9}$$

Next, we will estimate I_2. We can get, by splitting I_2 into two parts,

$$\begin{aligned} I_2 &= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i b_k\cdot\partial_k b_j\cdot\partial_i u_j dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_3 b_k\cdot\partial_k b_j\cdot\partial_3 u_j dx \\ &\equiv I_{21} + I_{22}. \end{aligned} \tag{2.10}$$

Now we go to estimating the first term in (2.10):

$$\begin{aligned} |I_{21}| &\leq \|\nabla_h u\|_{\gamma_1}\|\nabla b\|^2_{\frac{2\gamma_1}{\gamma_1-1}} \leq C\|\nabla_h u\|_{\gamma_1}\|\nabla b\|_2^{\frac{2\gamma_1-3}{\gamma_1}}\|\Delta b\|_2^{\frac{3}{\gamma_1}} \\ &\leq \frac{1}{16}\|\Delta b\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla b\|_2^2. \end{aligned}$$

Similarly, in view of Young's inequality, I_{22} yields

$$\begin{aligned} |I_{22}| &\leq \|\partial_3 b\|_{\gamma_2}\|\nabla b\|_{\frac{2\gamma_2}{\gamma_2-1}}\|\nabla u\|_{\frac{2\gamma_2}{\gamma_2-1}} \\ &\leq C\|\partial_3 b\|_{\gamma_2}\|\nabla u\|_2^{\frac{2\gamma_2-3}{2\gamma_2}}\|\Delta u\|_2^{\frac{3}{2\gamma_2}}\|\nabla b\|_2^{\frac{2\gamma_2-3}{2\gamma_2}}\|\Delta b\|_2^{\frac{3}{2\gamma_2}} \\ &\leq \frac{1}{8}\|\Delta u\|_2^2 + \frac{1}{16}\|\Delta b\|_2^2 + C\|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\|\nabla u\|_2\|\nabla b\|_2 \\ &\leq \frac{1}{8}\|\Delta u\|_2^2 + \frac{1}{16}\|\Delta b\|_2^2 + C\|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right). \end{aligned}$$

Inserting the above two inequalities into (2.10), we know that

$$|I_2| \leq \frac{1}{8}\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) + C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right). \tag{2.11}$$

Applying the same method, the following inequalities hold:

$$|I_3| \leq \frac{1}{8}\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) + C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) \tag{2.12}$$

$$|I_4| \leq \frac{1}{8}\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) + C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) \tag{2.13}$$

Putting (2.9) and (2.11)–(2.13) into (2.4), we obtain

$$\frac{d}{dt}\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right) + \left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) \leq C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\partial_3 b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right).$$

Note that

$$0 < \frac{2\gamma_i}{2\gamma_i - 3} \leq \alpha_i, \quad \text{if } \frac{2}{\alpha_i} + \frac{3}{\gamma_i} \leq 2, \quad i = 1, 2.$$

Therefore, by the standard Gronwall inequality, one has

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\nabla u(\cdot,t)\|_2^2 + \|\nabla b(\cdot,t)\|_2^2\right) + \int_0^T \left(\|\Delta u(\cdot,t)\|_2^2 + \|\Delta b(\cdot,t)\|_2^2\right)dt \\ &\quad \leq \left(\|\nabla u_0\|_2^2 + \|\nabla b_0\|_2^2\right)\exp\left(\int_0^T \left(\|\nabla_h u\|_{\gamma_1}^{\alpha_1} + \|\partial_3 b\|_{\gamma_2}^{\alpha_2}\right)dt\right).\end{aligned}$$

If $\gamma_1 = \gamma_2 = \infty$, we only need to use Hölder's inequality to obtain

$$|I_1| + |I_2| + |I_3| + |I_4| \leq C\left(\|\nabla_h u\|_\infty + \|\partial_3 b\|_\infty\right)\left(\|\nabla u_0\|_2^2 + \|\nabla b_0\|_2^2\right). \tag{2.14}$$

Combining (2.4) and (2.14) and Gronwall's inequality, we can get (2.1).

The last case is that where both $\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}}$ and $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}$ are sufficiently small. It follows that

$$\begin{aligned}|I_1| + |I_2| + |I_3| + |I_4| &\leq C\left(\|\nabla_h u\|_{\frac{3}{2}} + \|\partial_3 b\|_{\frac{3}{2}}\right)\left(\|\nabla u\|_6^2 + \|\nabla b\|_6^2\right) \\ &\leq C\left(\|\nabla_h u\|_{\frac{3}{2}} + \|\partial_3 b\|_{\frac{3}{2}}\right)\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right),\end{aligned} \tag{2.15}$$

where we have used $\|\nabla u\|_6 \leq C\|\Delta u\|_2$, since $u \in H^s, s > 3$. Combining (2.4) and (2.15) after integrating over $[0, T]$, we have

$$\begin{aligned}&\sup_{0\leq t\leq T}\left(\|\nabla u(\cdot,t)\|_2^2 + \|\nabla b(\cdot,t)\|_2^2\right) + 2\int_0^T \left(\|\Delta u(\cdot,t)\|_2^2 + \|\Delta b(\cdot,t)\|_2^2\right)dt \\ &\quad \leq \left(\|\nabla u_0\|_2^2 + \|\nabla b_0\|_2^2\right) + C\left(\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}} + \|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}\right)\int_0^T \left(\|\Delta u(\cdot,t)\|_2^2 + \|\Delta b(\cdot,t)\|_2^2\right)dt.\end{aligned}$$

So if $\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}}$ and $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}$ are sufficiently small, say $\|\nabla_h u\|_{L^{\infty,\frac{3}{2}}} \leq 1/C_1$ and $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}} \leq 1/C_2$, then (2.1) is true.

Consequently, this completes the proof of Theorem 1.1. □

3. Proof for Theorem 1.2

In this section, we will prove Theorem 1.2, which is different from the first theorem.

Proof. Now let us begin with (2.4), in which we can get the same estimate for I_1 as before, i.e., inequality (2.9) holds true. Next, we consider the other terms in different ways.

First, for I_2, we have

$$\begin{aligned}I_2 &= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k b_j \cdot \partial_i u_j dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k b_j \cdot \partial_3 u_j dx \\ &= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k b_j \cdot \partial_i u_j dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k b_j \cdot \partial_3 u_j dx + \sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_3 \cdot \partial_3 b_j \cdot \partial_3 u_j dx\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k b_j \cdot \partial_i u_j dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k b_j \cdot \partial_3 u_j dx - \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_i b_i \cdot \partial_3 b_j \cdot \partial_3 u_j dx \\
&\equiv I_{21} + I_{22} + I_{23}. \qquad (3.1)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|I_{21}| &= \|\nabla_h u\|_{\gamma_1} \|\nabla b\|^2_{\frac{2\gamma_1}{\gamma_1-1}} \le \|\nabla_h u\|_{\gamma_1} \|\nabla b\|_2^{\frac{2\gamma_1-3}{\gamma_1}} \|\Delta b\|_2^{\frac{3}{\gamma_1}} \\
&\le \frac{1}{16}\|\Delta b\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} \|\nabla b\|_2^2,
\end{aligned}
$$

where we use the well-known Sobolev and Young inequalities. For the last two terms of (3.1), we can get

$$
\begin{aligned}
|I_{22} + I_{23}| &\le \|\nabla_h b\|_{\gamma_2} \|\nabla b\|_{\frac{2\gamma_2}{\gamma_2-1}} \|\nabla u\|_{\frac{2\gamma_2}{\gamma_2-1}} \\
&\le C\|\nabla_h b\|_{\gamma_2} \|\nabla u\|_2^{\frac{2\gamma_2-3}{2\gamma_2}} \|\Delta u\|_2^{\frac{3}{2\gamma_2}} \|\nabla b\|_2^{\frac{2\gamma_2-3}{2\gamma_2}} \|\Delta b\|_2^{\frac{3}{2\gamma_2}} \\
&\le \frac{1}{8}\|\Delta u\|_2^2 + \frac{1}{16}\|\Delta b\|_2^2 + C\|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} \left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right).
\end{aligned}
$$

Combining the above two estimates with (3.1), it follows that

$$
|I_2| \le \frac{1}{8}\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) + C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right). \qquad (3.2)
$$

By integration by parts, we have

$$
\begin{aligned}
I_3 &= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j dx - \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_k \cdot \partial_k b_j \cdot \partial_3 b_j dx \\
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j dx - \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_k \cdot \partial_k b_j \cdot \partial_3 b_j dx - \sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_3 \cdot \partial_3 b_j \cdot \partial_3 b_j dx \\
&= -\sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i u_k \cdot \partial_k b_j \cdot \partial_i b_j dx - \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 u_k \cdot \partial_k b_j \cdot \partial_3 b_j dx + \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_i u_i \cdot \partial_3 b_j \cdot \partial_3 b_j dx \\
&\equiv I_{31} + I_{32} + I_{33}. \qquad (3.3)
\end{aligned}
$$

Applying Young's inequality as before, we know that

$$
|I_{31} + I_{33}| \le \|\nabla_h u\|_{\gamma_1} \|\nabla b\|^2_{\frac{2\gamma_1}{\gamma_1-1}} \le \frac{1}{16}\|\Delta b\|_2^2 + C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} \|\nabla b\|_2^2,
$$

and

$$
|I_{32}| \le \|\nabla_h b\|_{\gamma_2} \|\nabla b\|_{\frac{2\gamma_2}{\gamma_2-1}} \|\nabla u\|_{\frac{2\gamma_2}{\gamma_2-1}} \le \frac{1}{8}\|\Delta u\|_2^2 + \frac{1}{16}\|\Delta b\|_2^2 + C\|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} \left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right).
$$

Putting all the inequalities above into (3.3) yields

$$
|I_3| \le \frac{1}{8}\left(\|\Delta u\|_2^2 + \|\Delta b\|_2^2\right) + C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}} + \|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2 + \|\nabla b\|_2^2\right). \qquad (3.4)
$$

Using the same method as for (3.3), the last term in (2.4) can be decomposed into three parts:

$$
\begin{aligned}
I_4 &= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j dx + \sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k u_j \cdot \partial_3 b_j dx \\
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k u_j \cdot \partial_3 b_j dx + \sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_3 \cdot \partial_3 u_j \cdot \partial_3 b_j dx \\
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3} \partial_i b_k \cdot \partial_k u_j \cdot \partial_i b_j dx + \sum_{k=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_3 b_k \cdot \partial_k u_j \cdot \partial_3 b_j dx - \sum_{i=1}^{2}\sum_{j=1}^{3}\int_{\mathbb{R}^3} \partial_i b_i \cdot \partial_3 u_j \cdot \partial_3 b_j dx \\
&\equiv I_{41} + I_{42} + I_{43}. \qquad (3.5)
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
|I_{41}+I_{43}| &\le \|\nabla_h b\|_{\gamma_2}\|\nabla b\|_{\frac{2\gamma_2}{\gamma_2-1}}\|\nabla u\|_{\frac{2\gamma_2}{\gamma_2-1}} \\
&\le \frac{1}{8}\|\Delta u\|_2^2+\frac{1}{16}\|\Delta b\|_2^2+C\|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right),
\end{aligned}
$$

and

$$
|I_{42}|\le \|\nabla_h u\|_{\gamma_1}\|\nabla b\|^2_{\frac{2\gamma_1}{\gamma_1-1}} \le \frac{1}{16}\|\Delta b\|_2^2+C\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}\|\nabla b\|_2^2,
$$

which imply that (3.5) leads to

$$
|I_4|\le \frac{1}{8}\left(\|\Delta u\|_2^2+\|\Delta b\|_2^2\right)+C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}+\|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right). \tag{3.6}
$$

Summarizing (2.9), (3.2), (3.4) and (3.6), we obtain

$$
\frac{d}{dt}\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right)+\left(\|\Delta u\|_2^2+\|\Delta b\|_2^2\right)\le C\left(\|\nabla_h u\|_{\gamma_1}^{\frac{2\gamma_1}{2\gamma_1-3}}+\|\nabla_h b\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla u\|_2^2+\|\nabla b\|_2^2\right).
$$

Using the following relations:

$$
0<\frac{2\gamma_i}{2\gamma_i-3}\le \alpha_i,\quad \text{if } \frac{2}{\alpha_i}+\frac{3}{\gamma_i}\le 2,\quad i=1,2,
$$

and in view of Gronwall's inequality, one obtains

$$
\begin{aligned}
&\sup_{0\le t\le T}\left(\|\nabla u(\cdot,t)\|_2^2+\|\nabla b(\cdot,t)\|_2^2\right)+\int_0^T\left(\|\Delta u(\cdot,t)\|_2^2+\|\Delta b(\cdot,t)\|_2^2\right)dt \\
&\quad\le \left(\|\nabla u_0\|_2^2+\|\nabla b_0\|_2^2\right)\exp\left(\int_0^T\left(\|\nabla_h u\|_{\gamma_1}^{\alpha_1}+\|\nabla_h b\|_{\gamma_2}^{\alpha_2}\right)dt\right).
\end{aligned}
$$

Additionally, for the cases $\gamma_1=\gamma_2=\infty$ and $\gamma_1=\gamma_2=\frac{3}{2}$, the estimates are the same as for Theorem 1.1. So we omit them here.

This completes the proof of Theorem 1.2. □

4. Proof for Theorem 1.3

In this section, applying Theorem 1.2, we will prove Theorem 1.3.

Before the proof, we denote $\Delta_2 f$ as $\sum_{i=1}^2 \frac{\partial^2 f}{\partial x_i^2}$.

Proof. Adding the inner products of $\Delta_2 u$ with the first equation in (1.1), and of $\Delta_2 b$ with the second equation, and, additionally, using the divergence free property, leads to

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla_h u\|_2^2+\|\nabla_h b\|_2^2\right)+\left(\|\nabla\nabla_h u\|_2^2+\|\nabla\nabla_h b\|_2^2\right) \\
&\quad=\int_{\mathbb{R}^3}\left(u\cdot\nabla u\cdot\Delta_2 u-b\cdot\nabla b\cdot\Delta_2 u+u\cdot\nabla b\cdot\Delta_2 b-b\cdot\nabla u\cdot\Delta_2 b\right)dx \\
&\quad\equiv J_1+J_2+J_3+J_4.
\end{aligned} \tag{4.1}
$$

Step 1. The estimate for J_1.

Firstly, we can get

$$
\begin{aligned}
J_1 &= \sum_{i,j=1}^2\int_{\mathbb{R}^3} u_i\cdot\partial_i u_j\cdot\Delta_2 u_j dx+\sum_{i=1}^3\int_{\mathbb{R}^3} u_i\cdot\partial_i u_3\cdot\Delta_2 u_3 dx+\sum_{j=1}^2\int_{\mathbb{R}^3} u_3\cdot\partial_3 u_j\cdot\Delta_2 u_j dx \\
&\equiv J_{11}+J_{12}+J_{13}.
\end{aligned} \tag{4.2}
$$

By integration by parts, we have

$$\begin{aligned} J_{11} &= \frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_j u_i\cdot\partial_j u_i dx-\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_1 u_1\cdot\partial_2 u_2 dx+\int_{\mathbb{R}^3}\partial_3 u_3\cdot\partial_2 u_1\cdot\partial_1 u_2 dx\\ &= -\sum_{i,j=1}^{2} u_3\cdot\partial_3\partial_j u_i\cdot\partial_j u_i dx+\int_{\mathbb{R}^3} u_3\left(\partial_1\partial_3 u_1\cdot\partial_2 u_2+\partial_2\partial_3 u_2\cdot\partial_1 u_1\right)dx\\ &\quad -\int_{\mathbb{R}^3} u_3\left(\partial_2\partial_3 u_1\cdot\partial_1 u_2+\partial_1\partial_3 u_2\cdot\partial_2 u_1\right)dx. \end{aligned}$$

This expresses the well-known fact that for $u\in W^{1,2}_{\mathrm{div}}(\mathbb{R}^2)\cap W^{2,2}(\mathbb{R}^2)$, $\int_{\mathbb{R}^2}(u\cdot\nabla u)\cdot\Delta_2 u dx=0$. Furthermore, we have

$$J_{12}=-\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_i\cdot\partial_i u_3\cdot\partial_j u_3 dx=\sum_{i=1}^{3}\sum_{j=1}^{2}\int_{\mathbb{R}^3} u_3\cdot\partial_i\partial_j u_3\cdot\partial_j u_i dx,$$

which helps us to get

$$\begin{aligned} |J_{11}+J_{12}| &\le C\|u_3\|_{\gamma_1}\|\nabla\nabla_h u\|_2\|\nabla_h u\|_{\frac{2\gamma_1}{\gamma_1-2}}\le C\|u_3\|_{\gamma_1}\|\nabla\nabla_h u\|_2^{\frac{\gamma_1+3}{\gamma_1}}\|\nabla_h u\|_2^{\frac{\gamma_1-3}{\gamma_1}}\\ &\le \frac{1}{16}\|\nabla\nabla_h u\|_2^2+C\|u_3\|_{\gamma_1}^{\frac{2\gamma_1}{\gamma_1-3}}\|\nabla_h u\|_2^2, \end{aligned}\tag{4.3}$$

where we use the well-known Sobolev and Young inequalities. Now, we can obtain the following estimates for J_{13}:

$$\begin{aligned} J_{13} &= -\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_i u_3\cdot\partial_3 u_j\cdot\partial_i u_j dx-\sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_3\cdot\partial_i\partial_3 u_j\cdot\partial_i u_j dx\\ &= -\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_i u_3\cdot\partial_3 u_j\cdot\partial_i u_j dx+\frac{1}{2}\sum_{i,j=1}^{2}\int_{\mathbb{R}^3}\partial_3 u_3\cdot\left(\partial_i u_j\right)^2 dx\\ &\equiv J_{131}+J_{132}. \end{aligned}\tag{4.4}$$

For the first term in (4.4), we can estimate as follows:

$$\begin{aligned} |J_{131}| &\le \|\partial_3 u\|_{\gamma_2}\|\nabla_h u\|^2_{\frac{2\gamma_2}{\gamma_2-1}}\le C\|\partial_3 u\|_{\gamma_2}\|\nabla_h u\|_2^{\frac{2\gamma_2-3}{\gamma_2}}\|\nabla\nabla_h u\|_2^{\frac{3}{\gamma_2}}\\ &\le \frac{1}{32}\|\nabla\nabla_h u\|_2^2+C\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\|\nabla_h u\|_2^2. \end{aligned}$$

Using the same method, one gets the following inequalities for the second term:

$$|J_{132}|\le\|\partial_3 u_3\|_{\gamma_2}\|\nabla_h u\|^2_{\frac{2\gamma_2}{\gamma_2-1}}\le\frac{1}{32}\|\nabla\nabla_h u\|_2^2+C\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\|\nabla_h u\|_2^2.$$

Inserting the above two inequalities into (4.4), we have

$$|J_{13}|\le\frac{1}{16}\|\nabla\nabla_h u\|_2^2+C\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\|\nabla_h u\|_2^2.\tag{4.5}$$

Combining (4.2), (4.3) and (4.5), it holds that

$$|J_1|\le\frac{1}{8}\|\nabla\nabla_h u\|_2^2+C\left(\|u_3\|_{\gamma_1}^{\frac{2\gamma_1}{\gamma_1-3}}+\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\|\nabla_h u\|_2^2.\tag{4.6}$$

Step 2. The estimate for J_2.

Using the same method as for (4.2), we also can decompose J_2 into three parts:

$$\begin{aligned} -J_2 &= \sum_{i,j=1}^{2}\int_{\mathbb{R}^3} b_i\cdot\partial_i b_j\cdot\Delta_2 u_j dx+\sum_{i=1}^{3}\int_{\mathbb{R}^3} b_i\cdot\partial_i b_3\cdot\Delta_2 u_3 dx+\sum_{j=1}^{2}\int_{\mathbb{R}^3} b_3\cdot\partial_3 b_j\cdot\Delta_2 u_j dx\\ &\equiv J_{21}+J_{22}+J_{23}. \end{aligned}\tag{4.7}$$

For the first term J_{21}, after integration by parts, we can get

$$\begin{aligned} J_{21} &= -\int_{\mathbb{R}^3} \partial_2 b_2 \cdot \partial_1 b_1 \cdot \partial_3 u_3 dx + \int_{\mathbb{R}^3} \partial_1 b_2 \cdot \partial_2 b_1 \cdot \partial_3 u_3 dx + \sum_{i,j,k=1}^{2} \int_{\mathbb{R}^3} b_i \cdot \partial_k b_j \cdot \partial_i \partial_k u_j dx \\ &\equiv J_{211} + J_{212} + J_{213}, \end{aligned}$$

which implies that

$$\begin{aligned} |J_{211} + J_{212}| &\le \|\partial_3 u\|_{\gamma_2} \|\nabla_h b\|^2_{\frac{2\gamma_2}{\gamma_2 - 1}} \le C \|\partial_3 u\|_{\gamma_2} \|\nabla_h b\|_2^{\frac{2\gamma_2-3}{\gamma_2}} \|\nabla \nabla_h b\|_2^{\frac{3}{\gamma_2}} \\ &\le \frac{1}{48} \|\nabla\nabla_h b\|_2^2 + C \|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} \|\nabla_h b\|_2^2, \end{aligned}$$

and

$$\begin{aligned} |J_{213}| &\le \|b\|_{\gamma_3} \|\nabla\nabla_h u\|_2 \|\nabla_h b\|_{\frac{2\gamma_3}{\gamma_3-2}} \le \|b\|_{\gamma_3} \|\nabla\nabla_h u\|_2 \|\nabla_h b\|_2^{\frac{\gamma_3-3}{\gamma_3}} \|\nabla\nabla_h b\|_2^{\frac{3}{\gamma_3}} \\ &\le \frac{1}{24} \|\nabla\nabla_h u\|_2^2 + \frac{1}{48} \|\nabla\nabla_h b\|_2^2 + C \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} \|\nabla_h b\|_2^2. \end{aligned}$$

So finally we get, using the above inequalities,

$$|J_{21}| \le \frac{1}{24} \left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2 \right) + C \left(\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} + \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} \right) \|\nabla_h b\|_2^2. \tag{4.8}$$

By computation similar to that above, we get

$$J_{22} = -\sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_j b_i \cdot \partial_i b_3 \cdot \partial_j u_3 dx = \sum_{i=1}^{3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} u_3 \cdot \partial_i \partial_j b_3 \cdot \partial_j b_i dx.$$

Therefore, it follows that

$$\begin{aligned} |J_{22}| &\le \|u_3\|_{\gamma_1} \|\nabla\nabla_h b\|_2 \|\nabla_h b\|_{\frac{2\gamma_1}{\gamma_1-2}} \le C \|u_3\|_{\gamma_1} \|\nabla\nabla_h b\|_2^{\frac{\gamma_1+3}{\gamma_1}} \|\nabla_h b\|_2^{\frac{\gamma_1-3}{\gamma_1}} \\ &\le \frac{1}{24} \|\nabla\nabla_h b\|_2^2 + C \|u_3\|_{\gamma_1}^{\frac{2\gamma_1}{\gamma_1-3}} \|\nabla_h u\|_2^2. \end{aligned} \tag{4.9}$$

On the other hand, returning to J_{23} and integrating by parts, then we obtain

$$\begin{aligned} J_{23} &= -\sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_i b_3 \cdot \partial_3 b_j \cdot \partial_i u_j dx - \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} b_3 \cdot \partial_i \partial_3 b_j \cdot \partial_i u_j dx \\ &= -\sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_i b_3 \cdot \partial_3 b_j \cdot \partial_i u_j dx + \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} \partial_3 b_3 \cdot \partial_i b_j \cdot \partial_i u_j dx + \sum_{i,j=1}^{2} \int_{\mathbb{R}^3} b_3 \cdot \partial_i b_j \cdot \partial_3 \partial_i u_j dx \\ &\equiv J_{231} + J_{232} + J_{233}. \end{aligned}$$

So we get

$$\begin{aligned} |J_{231} + J_{232}| &\le \|\partial_3 b\|_{\gamma_4} \|\nabla_h u\|_{\frac{2\gamma_4}{\gamma_4-1}} \|\nabla_h b\|_{\frac{2\gamma_4}{\gamma_4-1}} \\ &\le \frac{1}{48} \left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2 \right) + C \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}} \left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2 \right), \end{aligned}$$

and

$$\begin{aligned} |J_{233}| &\le \|b\|_{\gamma_3} \|\nabla\nabla_h u\|_2 \|\nabla_h b\|_{\frac{2\gamma_3}{\gamma_3-2}} \\ &\le \frac{1}{48} \left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2 \right) + C \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} \|\nabla_h b\|_2^2. \end{aligned}$$

Summarizing the above two inequalities, we obtain

$$|J_{23}| \le \frac{1}{24}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \tag{4.10}$$

Inserting (4.8)–(4.10) into (4.7) yields

$$\begin{aligned} |J_2| \le\ & \frac{1}{8}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|u_3\|_{\gamma_1}^{\frac{2\gamma_1}{\gamma_1-3}} + \|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right. \\ & \left. + \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right) \times \left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \end{aligned} \tag{4.11}$$

Step 3. The estimate for J_3.

First, we divide J_3 as before:

$$\begin{aligned} J_3 =& \sum_{i,j=1}^{2}\int_{\mathbb{R}^3} u_i \cdot \partial_i b_j \cdot \Delta_2 b_j dx + \sum_{i=1}^{3}\int_{\mathbb{R}^3} u_i \cdot \partial_i b_3 \cdot \Delta_2 b_3 dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} u_3 \cdot \partial_3 b_j \cdot \Delta_2 b_j dx \\ \equiv& J_{31} + J_{32} + J_{33}. \end{aligned}$$

Using the same method, we can estimate parts one by one, i.e.,

$$|J_{31} + J_{33}| \le \frac{1}{16}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right),$$

and

$$|J_{32}| \le \frac{1}{16}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right).$$

Therefore,

$$|J_3| \le \frac{1}{8}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} + \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right) \times \left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \tag{4.12}$$

Step 4. The estimate for J_4.

Now, we will use a similar estimate for J_4:

$$\begin{aligned} -J_4 =& \sum_{i,j=1}^{2}\int_{\mathbb{R}^3} b_i \cdot \partial_i u_j \cdot \Delta_2 b_j dx + \sum_{i=1}^{3}\int_{\mathbb{R}^3} b_i \cdot \partial_i u_3 \cdot \Delta_2 b_3 dx + \sum_{j=1}^{2}\int_{\mathbb{R}^3} b_3 \cdot \partial_3 u_j \cdot \Delta_2 b_j dx \\ \equiv& J_{41} + J_{42} + J_{43}. \end{aligned}$$

Similarly, after estimating each term, it follows that

$$|J_{41}| \le \frac{1}{16}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right),$$

$$|J_{42} + J_{43}| \le \frac{1}{16}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}}\|\nabla_h u\|_2^2,$$

which implies that

$$|J_4| \le \frac{1}{8}\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) + C\left(\|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \tag{4.13}$$

Putting (4.6) and (4.11)–(4.13) into (4.1) yields

$$\begin{aligned} & \frac{d}{dt}\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right) + \left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right) \\ & \le C\left(\|u_3\|_{\gamma_1}^{\frac{2\gamma_1}{\gamma_1-3}} + \|\partial_3 u\|_{\gamma_2}^{\frac{2\gamma_2}{2\gamma_2-3}} + \|b\|_{\gamma_3}^{\frac{2\gamma_3}{\gamma_3-3}} + \|\partial_3 b\|_{\gamma_4}^{\frac{2\gamma_4}{2\gamma_4-3}}\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \end{aligned}$$

Note that

$$0 < \frac{2\gamma_i}{\gamma_i - 3} \le \alpha_i, \quad \text{if } \frac{2}{\alpha_i} + \frac{3}{\gamma_i} \le 1, \quad i = 1, 3,$$
$$0 < \frac{2\gamma_j}{2\gamma_j - 3} \le \alpha_j, \quad \text{if } \frac{2}{\alpha_j} + \frac{3}{\gamma_j} \le 2, \quad j = 2, 4.$$

Just as the proof of the former theorems, Gronwall's inequality yields

$$\begin{aligned}
&\sup_{0\le t\le T} \left(\|\nabla_h u(\cdot,t)\|_2^2 + \|\nabla_h b(\cdot,t)\|_2^2\right) + \int_0^T \left(\|\nabla\nabla_h u(\cdot,t)\|_2^2 + \|\nabla\nabla_h b(\cdot,t)\|_2^2\right) dt \\
&\quad \le \left(\|\nabla_h u_0\|_2^2 + \|\nabla_h b_0\|_2^2\right) \exp\left(\int_0^T \left(\|u_3\|_{\gamma_1}^{\alpha_1} + \|\partial_3 u\|_{\gamma_2}^{\alpha_2} + \|b\|_{\gamma_3}^{\alpha_3} + \|\partial_3 b\|_{\gamma_4}^{\alpha_4}\right) dt\right),
\end{aligned}$$

which implies that $\nabla_h u, \nabla_h b \in L^\infty\left(0, T; L^2\right)$ holds. Then the proof is complete due to Theorem 1.2.

If $\gamma_i = \infty$, $i = 1, 2, 3, 4$, the case can be treated similarly to the limit case of Theorem 1.1 for $\gamma_1 = \gamma_2 = \infty$. Thanks to Hölder's inequality,

$$|J_1| + |J_2| + |J_3| + |J_4| \le C\left(\|u_3\|_\infty + \|\partial_3 u\|_\infty + \|b\|_\infty + \|\partial_3 b\|_\infty\right)\left(\|\nabla_h u\|_2^2 + \|\nabla_h b\|_2^2\right). \tag{4.14}$$

Putting (4.14) into (4.1), Gronwall's inequality also yields $\nabla_h u, \nabla_h b \in L^\infty\left(0, T; L^2\right)$.

If $\gamma_1 = \gamma_3 = 3$ and $\gamma_2 = \gamma_4 = \frac{3}{2}$, thanks to Sobolev embedding, we have

$$\begin{aligned}
|J_1| + |J_2| + |J_3| + |J_4| &\le C\left(\|u_3\|_3 + \|\partial_3 u\|_{\frac{3}{2}} + \|b\|_3 + \|\partial_3 b\|_{\frac{3}{2}}\right)\left(\|\nabla_h u\|_6^2 + \|\nabla_h b\|_6^2\right) \\
&\le C\left(\|u_3\|_3 + \|\partial_3 u\|_{\frac{3}{2}} + \|b\|_3 + \|\partial_3 b\|_{\frac{3}{2}}\right)\left(\|\nabla\nabla_h u\|_2^2 + \|\nabla\nabla_h b\|_2^2\right).
\end{aligned}$$

Then, $\nabla_h u, \nabla_h b \in L^\infty\left(0, T; L^2\right)$ follows from the smallness of $\|u_3\|_{L^{\infty,3}}$, $\|\partial_3 u\|_{L^{\infty,\frac{3}{2}}}$, $\|b\|_{L^{\infty,3}}$ and $\|\partial_3 b\|_{L^{\infty,\frac{3}{2}}}$.

The proof is finished. □

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